RamSeries theory manual

Compass Ingeniería y Sistemas – http://www.compassis.com
Tel:+34 932 181 989 – E:info@compassis.com – C/Tuset 8, 7o 2a, Barcelona 08006 (Spain)

Table of contents

1. Introduction

RamSeries is a numerical simulation solver for structural analysis, based on the finite element method (FEM). It offers a large set of numerical capabilities that can be used for the analysis of a wide range of structural problems, from linear statics to linear and non-linear transient simulations and modal analysis. It can be used as well to simulate impact, coupled fluid-structure analysis, thermomechanical problems, fatigue assessment of structures and much more. RamSeries includes a full range of 3D structural element models, including beams, cables, shells, membranes and solid elements. It also considers both linear and non-linear material's constitutive laws including advanced models for laminated composite structures.

This document is intended to provide a brief introduction to some of the basic theoretical aspects underlying the RamSeries solver. To this aim, the fundamentals of structural analysis are presented first together with the basic material constitutive models available in RamSeries. Next, fundamental aspects of the dynamic analysis and the non-linear analysis are also underlined.

2. Structural analysis and material's constitutive models

In this chapter, the fundamental theoretical aspects of the structural analysis of three-dimensional solids and shells are presented in sections 2.1 and 2.2 respectively. Within this context, linear isotropic and orthotropic material constitutive models are described. Next, the classic lamination theory implemented in RamSeries is described in section 2.3. The more advanced serial-parallel theory, suitable for the analysis of generalized composite materials, is described in section 2.4. Finally, to close the chapter, the J2-flow and the isotropic damage non-linear constitutive models are presented in sections 2.6 and 2.7.

2.1. Analysis of three-dimensional solids

In this section, a summary of the theory of 3D solids analysis is provided. The basic assumptions under the formulation implemented within RamSeries are as follows:

1. Small displacements
2. Linear elasticity of materials
3. The principle of loads superposition

For a 3D solid, the displacement of a given point is completely determined by the values of the three degrees of freedom u, v, w.

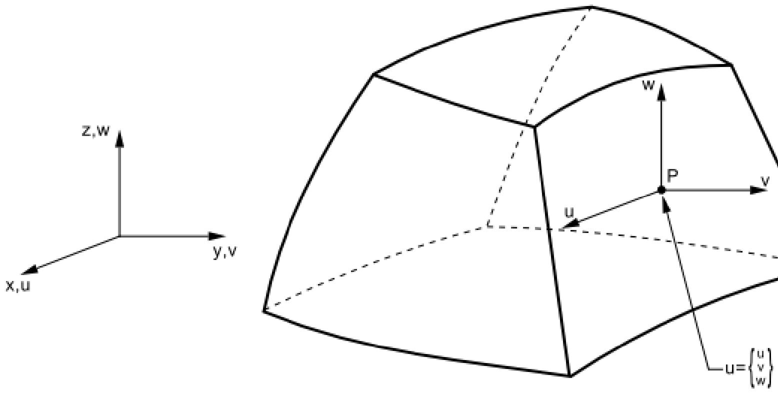

Figure 2-1. Schematic representation of a 3D solid. The deformation of the solid is completely determined once the displacement field (u, v, w) is known. Hence, three degrees of freedom must be considered for each material point.

Following the three-dimensional theory of elasticity, the deformation of the solid is defined, using the Voigt notation, as follows:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{bmatrix} \quad \text{Eq. 2-1}$$

In the most general case of anisotropic elasticity, the stress-strain relation is provided through a symmetric 6x6 constitutive matrix with 21 independent coefficients. Nevertheless, the more simple and commonly used orthotropic case is considered in RamSeries. Hence, considering x', y', z' the principal orthotropic directions of the solid material, the constitutive equation in local axes can be written as:

$$\begin{bmatrix} \epsilon'_x \\ \epsilon'_y \\ \epsilon'_z \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{x'}} & -\frac{\nu_{y'x'}}{E_{y'}} & -\frac{\nu_{z'x'}}{E_{z'}} & 0 & 0 & 0 \\ -\frac{\nu_{x'y'}}{E_{x'}} & \frac{1}{E_{y'}} & -\frac{\nu_{z'y'}}{E_{z'}} & 0 & 0 & 0 \\ -\frac{\nu_{x'z'}}{E_{x'}} & -\frac{\nu_{y'z'}}{E_{y'}} & \frac{1}{E_{z'}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{x'y'}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{x'z'}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{y'z'}} \end{bmatrix} \begin{bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{xz} \\ \tau'_{yz} \end{bmatrix} \qquad \text{Eq. 2-2}$$

where only 9 independent material constants actually remain since the following relations must hold due to the symmetry of the constitutive matrix:

$$E_{x'}\nu_{y'x'} = E_{y'}\nu_{x'y'} \; ; \; E_{y'}\nu_{z'y'} = E_{z'}\nu_{y'z'} \; ; \; E_{z'}\nu_{x'z'} = E_{x'}\nu_{z'x'} \qquad \text{Eq. 2-3}$$

In the isotropic case only two material parameters, the Young's modulus E and the Poisson coefficient ν, are required and the constitutive relationreduces to:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \boldsymbol{D} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \qquad \text{Eq. 2-4}$$

where the stiffness matrix $\boldsymbol{D}$ is given by:

$$D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \cdot \\ \cdot \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & Symm & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \qquad \text{Eq. 2-5}$$

2.2. Analysis of shells

Glossary

Symbol	Description	SI units
x', y', z'	Local shell coordinate directions	m
u', v', w'	Displacements of a generic point in the local coordinate system of the shell	m
u'_o, v'_o, w'_o	Displacements of a mid-plane point of the shell in the local coordinate system of the shell	m
$\theta_{x'}, \theta_{y'}$	Rotations in the local coordinate system of the shell	rad
ϵ'_x, ϵ'_y	In-plane shell axial strains	-
$\gamma_{x'y'}$	In-plane shell shear strain	-
$\gamma_{x'z'}, \gamma_{y'z'}$	Transverse shell shear strains	-
$\sigma_{x'}, \sigma_{y'}$	In-plane shell axial stresses	Pa
$\tau_{x'y'}$	In-plane shell shear stress	Pa
$\tau_{x'z'}, \tau_{y'z'}$	Transverse shell shear stresses	Pa
$\boldsymbol{D}_f{}'$	Flexural constitutive matrix	Pa
$\boldsymbol{D}_s{}'$	Shear constitutive matrix	Pa
α	Shear correction factor	-
$E_{x'}, E_{y'}$	Young modulus in the principal in-plane directions of the shell	Pa
$G_{x'y'}, G_{y'x'}$	Shell's in-plane shear modulus	Pa
$\nu_{x'y'}, \nu_{y'x'}$	Shell's in-plane Poisson coefficients	-
N_i	Normal forces per unit length (membrane forces)	N/m
N_{ij}	Horizontal shear forces per unit length (membrane forces)	N/m
M_i	Bending moments per unit length	N
V_i	Shear forces per unit length	N/m
M_{ij}	Twisting moment per unit length	N
$\hat{\boldsymbol{\sigma}}'_m$	Generalized local membrane stresses of the shell	N/m
$\hat{\boldsymbol{\sigma}}'_f$	Generalized local flexural stresses of the shell	N
$\hat{\boldsymbol{\sigma}}'_s$	Shear generalized local stresses of the shell	N/m

The analysis of shells is based on approximating a real solid by its corresponding medium plane, assuming one of its dimensions is much smaller than the other two. Several additional assumptions and simplifications are also necessary to close the model. In particular, the Reissner-Mindlin hypothesis is adopted in RamSeries for the implementation of a structural model for shells. These hypothesis can be summarized as follows:

1. Small displacements
2. Linear elasticity of materials
3. Principle of loads superposition
4. Accounting for shear deformations
5. All the points belonging to a given normal to the medium plane have the same vertical displacement (in the z' local shell's direction)

6. All points that, before the deformation, belonged to a given normal to the medium plane, after deformation continue belonging to a straight line, no longer necessarily orthogonal to the deformed medium plane
7. Normal stress $\sigma_z$is neglected

Only assumptions 5, 6 and 7 are specific to the Reissner-Mindlin hypothesis. The remaining ones apply to both, the Reissner-Mindlin and the Kirchhoff shell models.

From the assumptions above it is possible to deduce the following expression for the shell's deformation [1]:

$$\begin{bmatrix} \epsilon_x{}' \\ \epsilon_y{}' \\ \gamma_{x'y'} \\ \dots \\ \gamma_{x'z'} \\ \gamma_{y'z'} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_o'}{\partial x'} \\ \dfrac{\partial v_o'}{\partial y'} \\ \dfrac{\partial u_o'}{\partial y'} + \dfrac{\partial v_o'}{\partial x'} \\ \dots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -z' \dfrac{\partial \theta_{x'}}{\partial x'} \\ -z' \dfrac{\partial \theta_{y'}}{\partial y'} \\ -z' \left(\dfrac{\partial \theta_{x'}}{\partial y'} + \dfrac{\partial \theta_{y'}}{\partial x'} \right) \\ \dots \\ \dfrac{\partial w_o'}{\partial x'} - \theta_{x'} \\ \dfrac{\partial w_o'}{\partial y'} - \theta_{y'} \end{bmatrix} \qquad \text{Eq. 2-6}$$

where $u_o', v_o', w_o', \theta_{x'}, \theta_{y'}$ are the five degrees of freedom to be considered for each node of the shell. The ' symbol that appears in all the degrees of freedom and coordinates that take part in the definition of the strain tensor in Eq. 2-6, indicates that they are referred to the local coordinates system of the shell.

The shell's constitutive equation is given by:

$$\begin{bmatrix} \sigma_x{}' \\ \sigma_y{}' \\ \tau_{x'y'} \\ \dots \\ \tau_{x'z'} \\ \tau_{y'z'} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_f{}' & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \boldsymbol{D}_s{}' \end{bmatrix} \begin{bmatrix} \epsilon_x{}' \\ \epsilon_y{}' \\ \gamma_{x'y'} \\ \dots \\ \gamma_{x'z'} \\ \gamma_{y'z'} \end{bmatrix} \qquad \text{Eq. 2-7}$$

where $\boldsymbol{D}_f{}'$ and $\boldsymbol{D}_s'$ are the flexural and shear constitutive matrices respectively. These constitutive matrices can be written, for an orthotropic material, as follows:

$$\boldsymbol{D}_f{}' = \frac{1}{1 - \nu_{x'y'}\nu_{y'x'}} \begin{bmatrix} E_{x'} & \nu_{x'y'}E_{x'} & 0 \\ \nu_{y'x'}E_{y'} & E_{y'} & 0 \\ 0 & 0 & (1 - \nu_{x'y'}\nu_{y'x'})G_{x'y'} \end{bmatrix} \qquad \text{Eq. 2-8}$$

$$D'_s = \begin{bmatrix} \alpha G_{x'y'} & 0 \\ 0 & \alpha G_{y'x'} \end{bmatrix} \quad \text{Eq. 2-9}$$

The factor α in Eq. 2-9 is the coefficient to correct for the transversal tangential work (or shear correction factor).

In addition, due to the fact that for plates and shells the actual solid is approximated by the shell's mid-plane, instead of solving for the 3-dimensional stress distribution through the entire body, the problem must be actually solved for certain stress resultants. These stress resultants are line distributions of force or moment per unit length along curves located in the middle plane of the plate or shell. These line distributions are statically equivalent to the distributions of stress on certain planes of the 3-dimensional body [2] (see Figure 2-2 and Figure 2-3).

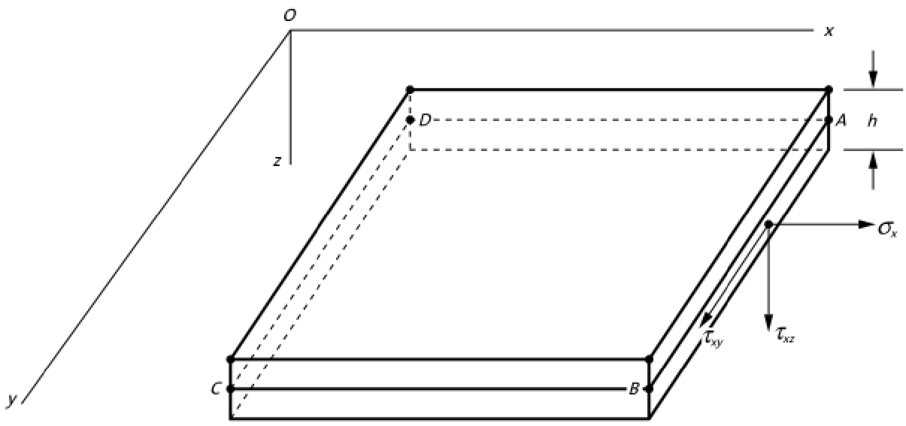

Figure 2-2. Schematic representation of a plate or shell element of thickness *h*. Within the context of a shell theory, the actual solid is approximated by its mid-plane (ABCD), and the stresses distributions (σ_x, τ_{xy}, τ_{xz}) are replaced by stress resultants along curves in the middle plane (AB for instance in the figure).

In this sense, the shell theory replaces the distributed normal stress σ_x per unit area on the positive x-face of the element by a normal force N_x per unit length along AB, and a bending moment M_x per unit length along AB that can be expressed as:

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz \quad \text{Eq. 2-10}$$

$$M_x = \int_{-h/2}^{h/2} z\sigma_x dz \quad \text{Eq. 2-11}$$

These line distributions of force and moment are statically equivalent to the surface distribution σ_x. Similarly, the distribution of vertical shear stress τ_{xz} is statically equivalent to a vertical shear force V_x per unit length along AB in the form:

$$V_x = \int_{-h/2}^{h/2} \tau_{xz} dz \qquad \text{Eq. 2-12}$$

And finally, the horizontal shear forces resulting from the shear stress distribution τ_{xz} are represented by a twisting moment per unit length of the form:

$$M_{xy} = -\int_{-h/2}^{h/2} z\tau_{xy} dz \qquad \text{Eq. 2-13}$$

Finally, the horizontal shear forces per unit length N_{xy} and N_{yx} are defined as:

$$N_{xy} = N_{yx} = \int_{-h/2}^{h/2} \tau_{xy} dz \qquad \text{Eq. 2-14}$$

Forces N_y and V_y and bending moments M_y per unit length are defined by replacing x by y in the equations above. Figure 2-3 summarizes the positive sign convention for the stress resultants.

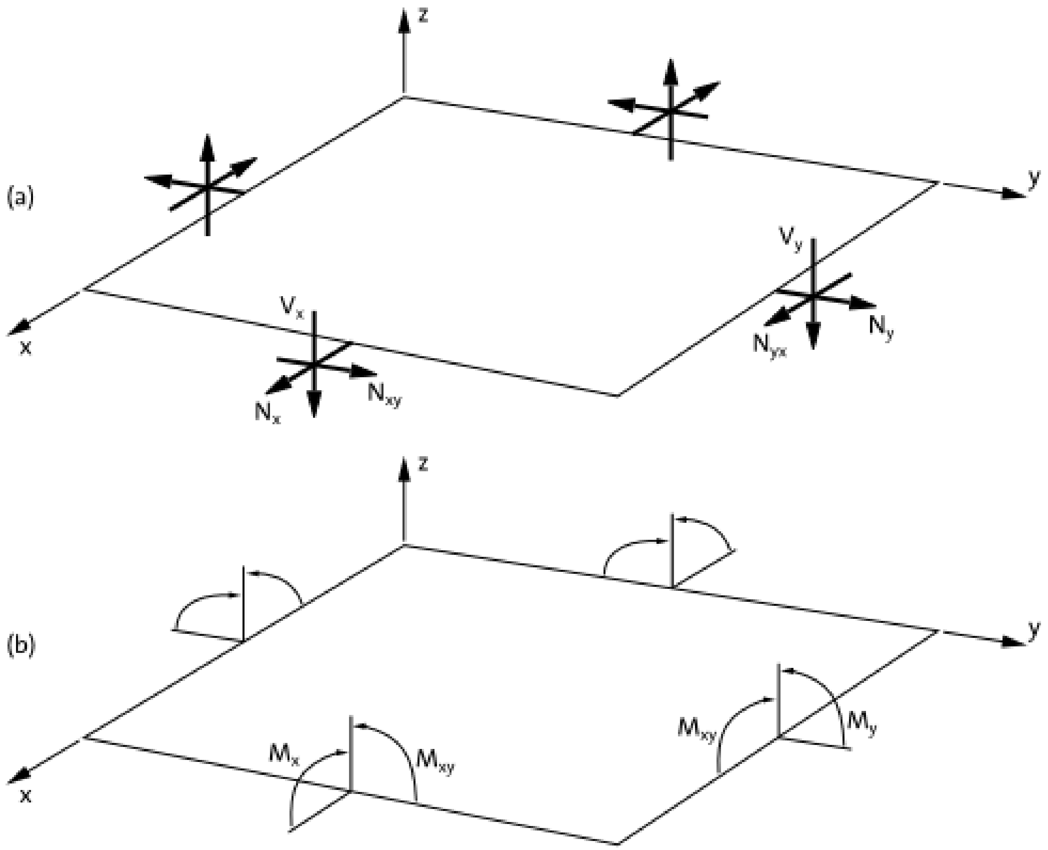

Figure 2-3. Sign convention for stress resultants. (a) Positive forces. (b) Positive moments.

The forces per unit length (N_x, N_y and N_{xy}) acting in the plane of the shell are called membrane forces.

Thus, summarizing it is possible to define the strengths on the shell as follows:

$$\widehat{\boldsymbol{\sigma}}' = \begin{bmatrix} \widehat{\boldsymbol{\sigma}}'_m \\ \cdots \\ \widehat{\boldsymbol{\sigma}}'_f \\ \cdots \\ \widehat{\boldsymbol{\sigma}}'_s \end{bmatrix} = \begin{bmatrix} N_x{}' \\ N_y{}' \\ N_{x'y'} \\ \cdots \\ M_{x'} \\ M_{y'} \\ M_{x'y'} \\ \cdots \\ V_{x'} \\ V_{y'} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \\ \cdots \\ z'\sigma_{x'} \\ z'\sigma_{y'} \\ z'\tau_{x'y'} \\ \cdots \\ \tau_{x'z'} \\ \tau_{y'z'} \end{bmatrix} dz' = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \boldsymbol{\sigma}_f{}' \\ \cdots \\ z'\boldsymbol{\sigma}_f{}' \\ \cdots \\ \boldsymbol{\sigma}_s{}' \end{bmatrix} dz' \qquad \text{Eq. 2-15}$$

where h is the thickness of the shell and $\widehat{\boldsymbol{\sigma}}'_m$, $\widehat{\boldsymbol{\sigma}}'_f$, and $\widehat{\boldsymbol{\sigma}}'_s$ are the membrane, flexural and shear generalized local stresses respectively. The relation between the generalized stresses and the generalized local deformations can be finally obtained from Eq. 2-7 and Eq. 2-15.

$$\widehat{\boldsymbol{\sigma}}' = \begin{bmatrix} \widehat{\boldsymbol{\sigma}}'_m \\ \cdots \\ \widehat{\boldsymbol{\sigma}}'_f \\ \cdots \\ \widehat{\boldsymbol{\sigma}}'_s \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \boldsymbol{\sigma}_f{}' \\ \cdots \\ z'\boldsymbol{\sigma}_f{}' \\ \cdots \\ \boldsymbol{\sigma}_s{}' \end{bmatrix} dz' = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \boldsymbol{D}'_f\left(\widehat{\boldsymbol{\epsilon}}'_m + z'\widehat{\boldsymbol{\epsilon}}'_f\right) \\ \cdots \\ z'\boldsymbol{D}'_f\left(\widehat{\boldsymbol{\epsilon}}'_m + z'\widehat{\boldsymbol{\epsilon}}'_f\right) \\ \cdots \\ \boldsymbol{D}'_s\widehat{\boldsymbol{\epsilon}}'_s \end{bmatrix} dz' = \widehat{\boldsymbol{D}}' \begin{bmatrix} \widehat{\boldsymbol{\epsilon}}'_m \\ \cdots \\ \widehat{\boldsymbol{\epsilon}}'_f \\ \cdots \\ \widehat{\boldsymbol{\epsilon}}'_s \end{bmatrix} \qquad \text{Eq. 2-16}$$

being $\widehat{\boldsymbol{D}}'$ the shell constitutive matrix.

$$\widehat{\boldsymbol{D}}' = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \boldsymbol{D}'_f & z'\boldsymbol{D}'_f & 0 \\ z'\boldsymbol{D}'_f & z'^2\boldsymbol{D}'_f & 0 \\ 0 & 0 & \boldsymbol{D}'_s \end{bmatrix} \qquad \text{Eq. 2-17}$$

2.3. Classical lamination theory

Glossary

Symbol	Description	Units
$[T_\sigma]$	Laminate axis to ply axis stress transformation matrix	-
$[T_\epsilon]$	Laminate axis to ply axis engineering strain transformation matrix	-
$[Q_{ij}]$	Ply modulus matrix	N/m^2
N	Number of plies	-
k	Ply number (1 to n)	-

z_k	Height to ply mid-thickness from laminate mid-thickness	m
$[A]$	Extensional stiffness matrix	N/m
$[B]$	Bending-extension coupling stiffness matrix	N
$[D]$	Bending stiffness matrix	N·m
$[H]$	Transverse shear stiffness matrix	N·m
(N_x, N_y, N_{xy})	Laminate in-plane (membrane) forces per unit width	N/m
(M_x, M_y, M_{xy})	Laminate bending moments per unit length	N
(V_y, V_x)	Transverse shear forces per unit length	N/m
$(\epsilon_x^0, \epsilon_y^0, \gamma_{xy}^0)$	Laminate in-plane strains	-
$(\gamma_{yz}, \gamma_{xz})$	Laminate transverse shear strains	-
ϕ_x, ϕ_y	Rotations in the local coordinate system of the shell	rad
$(\kappa_x, \kappa_y, \kappa_{xy})$	Laminate curvatures (due to bending and twisting)	m^{-1}

Failure criteria

FI	Failure index	-
R	Reserve factor (or strength ratio)	-
S_{t1}	Tensile strength in the fibre direction of the ply	N/m^2
S_{c1}	Compressive strength in the fibre direction of the ply	N/m^2
S_{t2}	Tensile strength in the transverse direction of the ply	N/m^2
S_{c2}	Compressive strength in the transverse direction of the ply	N/m^2
T_{12}	In-plane shear strength of the ply	N/m^2

The Classical Lamination Theory (CLT) can be used to determine the stiffness, the strength and the effective material properties of laminated shells, as it is explained in many textbooks [3]. The fundamental assumptions are:

- Laminate plies are perfectly bonded together.
- The bonds are thin and displacements are continuous across boundaries.
- Displacements are small compared to the laminate thickness.
- Strains are small compared to unity.
- The application of Kirchoff hypothesis for plates. In-plane displacements are a linear function of the thickness coordinates, which result in negligible interlaminar shear strain.
- The laminate thickness is small compared with the lateral dimensions.

The main objective of the CLT is to determine the laminate stiffness matrix, which is central for subsequent structural analysis. To this aim, first the constitutive behaviour of individual plies must be determined. Next, the relationship between the structural properties of the entire laminate and those of the individual plies and their corresponding orientations must be addressed.

2.3.1. Mechanics of individual plies (ply mechanics)

A laminate is a set of plies with different fibre orientations bonded together. Hence, before developing laminate properties, it is necessary to address the issue on how to transform stresses, strains, compliances and stiffness from the ply coordinate system to the laminate coordinate system. Finally, the modulus matrix, thickness and height to ply mid-thickness are used to sum the contribution of each ply to obtain the laminate stiffness matrix.

The stress transformation matrix used to transform stress from the laminate axis system to the ply axis system is given by:

$$[T_\sigma] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}$$ Eq. 2-18

where $m = \cos^2\theta$ and $n = \sin^2\theta$ and θ determines the orientation of the ply as shown in Figure 2-5.

On the other hand, the strain transformation matrix differs depending on whether the formulation at hand is dealing with tensorial strain or engineering strain. The Reuters transformation matrix can be used to cope effectively with the factors of one half and two in the stiffness and compliance matrices that arise depending on the actual strain being used [4].

$$[T_\epsilon] = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix}$$ Eq. 2-19

The ply modulus matrix in laminate system is finally obtained as:

$$[Q_{ij}]_{KL} = [T_\sigma]_k^{-1}[Q_{ij}]_{kp}[T_\epsilon]_k$$ Eq. 2-20

2.3.2. Mechanics of the entire laminate (macromechanics)

The basic building block of a composite structure is a plate element. Even composite beams are thin-walled sections composed of plate elements [4]. The constitutive equations for such an element are presented here.

First, it must be noted that the sign conventions used for the definition of the laminate stacking sequence, ply orientation, strains, curvature and applied membrane and bending loads affect the sign of some of the terms of the laminate stiffness matrix. The convention used in RamSeries is shown in the following figure.

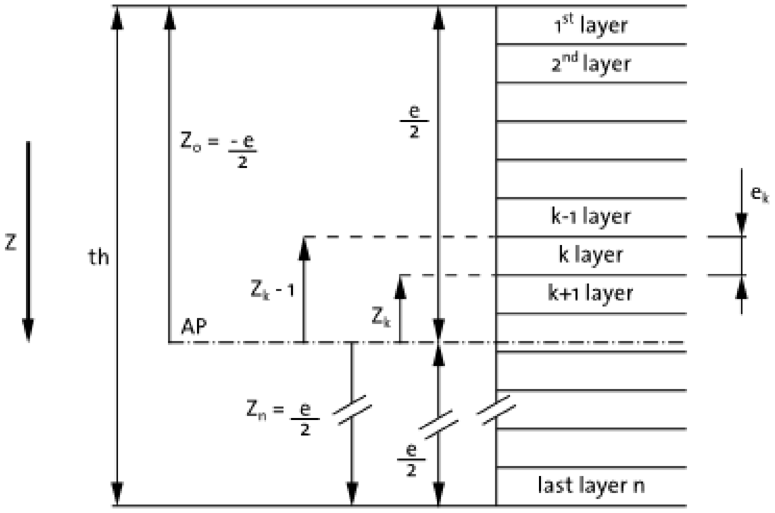

Figure 2-4. Laminate's stacking sequence definition and sign convention.

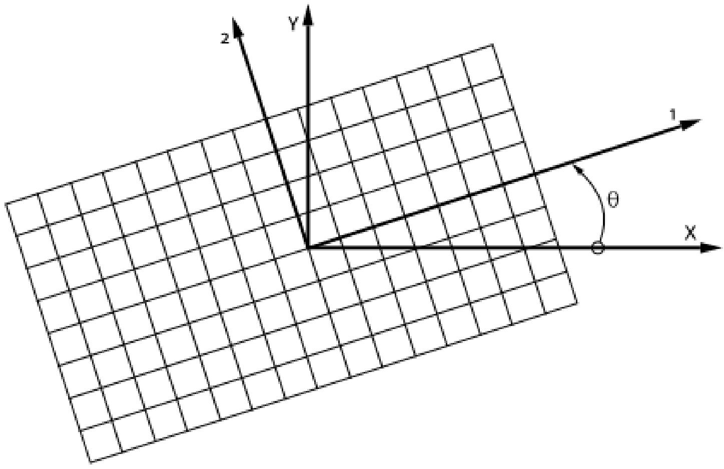

Figure 2-5. Orientation of individual ply local axes (1,2) in relation to the laminate global axes (x,y).

The basis of the first-order shear deformation theory (FSDT) of plates is given by the following equations:

$$\epsilon_x^0(x,y) = \frac{\partial u_0}{\partial x} \qquad \text{Eq. 2-21}$$

$$\epsilon_y^0(x,y) = \frac{\partial v_0}{\partial y} \qquad \text{Eq. 2-22}$$

$$\gamma_{xy}^0(x,y) = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \qquad \text{Eq. 2-23}$$

$$\kappa_x\ (x,y) = \frac{\partial \phi_x}{\partial x}$$ Eq. 2-24

$$\kappa_y\ (x,y) = \frac{\partial \phi_y}{\partial y}$$ Eq. 2-25

$$\kappa_{xy}(x,y) = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}$$ Eq. 2-26

$$\gamma_{yz}(x,y,z) = -\phi_y + \partial w_0/\partial y$$ Eq. 2-27

$$\gamma_{xz}(x,y,z) = -\phi_x + \partial w_0/\partial x$$ Eq. 2-28

On the other hand, the basis for the classical plate theory (CPT) is given by the following set of equations, where transverse shear deformations (γ_{xz} and γ_{yz}) are neglected:

$$\epsilon_x^0(x,y) = \frac{\partial u_0}{\partial x}$$ Eq. 2-29

$$\epsilon_y^0(x,y) = \frac{\partial v_0}{\partial y}$$ Eq. 2-30

$$\gamma_{xy}^0(x,y) = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$ Eq. 2-31

$$\kappa_x\ (x,y) = \frac{\partial^2 w_0}{\partial x^2}$$ Eq. 2-32

$$\kappa_y\ (x,y) = \frac{\partial^2 w_0}{\partial y^2}$$ Eq. 2-33

$$\kappa_{xy}(x,y) = 2\frac{\partial^2 w_0}{\partial x \partial y}$$ Eq. 2-34

FSDT is more accurate specially in the case of laminated composites since these materials have low shear modulus (G<E/10) thus requiring transverse shear deformations to be taken into account.
Up to this point, the strains at every point (x,y,z) of the plate have been replaced by the corresponding mid-surface strains ($\epsilon_x^0, \epsilon_y^0, \gamma_{xy}^0$) and the surface curvatures ($\kappa_x, \kappa_y, \kappa_{xy}$):

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$ Eq. 2-35

Integrating the stresses over the thickness of the plate it is possible to obtain the resultant forces and moments per unit width of laminate as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \qquad \text{Eq. 2-36}$$

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \cdot z \cdot dz$$

Note that with this definition, the moment is positive when the stress is compressive above the mid-surface and tensile below it. Also, positive curvature results in a deflection w(x,y) that is concave up, as it happens in a simply supported beam under its own weight.

For a laminate, the above defined integrals span over several plies. Therefore, the integrals can be divided into summations of integrals over each lamina.

The laminate stiffness matrix can be defined as follows:

$$\begin{Bmatrix} N \\ M \end{Bmatrix}_L = \begin{pmatrix} A & B \\ B & D \end{pmatrix}_L \begin{Bmatrix} \epsilon_{mp} \\ \kappa_{mp} \end{Bmatrix}_L \qquad \text{Eq. 2-37}$$

$$\{V\}_L = (H)_L \{\gamma_{mp}\}_L \qquad \text{Eq. 2-38}$$

where:

$$[A] = \sum_1^n [Q_{ij}]_k t_k \;\; ; i,j = 1,2,6 \qquad \text{Eq. 2-39}$$

$$[B] = \sum_1^n [Q_{ij}]_k t_k z_k \; ; i,j = 1,2,6 \qquad \text{Eq. 2-40}$$

$$[D] = \sum_1^n [Q_{ij}]_k \left(t_k z_k^2 + \left(\frac{t_k^3}{12} \right) \right) ; i,j = 1,2,6 \qquad \text{Eq. 2-41}$$

$$[H] = -\frac{5}{4} \sum_1^n [Q_{ij}]_k \left(t_k - \frac{4}{t^2} \left(t_k z_k^2 + \frac{t_k^3}{12} \right) \right) ; i,j = 4,5 \qquad \text{Eq. 2-42}$$

Note that $[A], [B], [D]$ are all 3x3 symmetric matrices, whose coefficients are functions of the thickness, orientation, stacking sequence, and material properties of the plies [4]. $[A]$ is the in-plane stiffness matrix that relates in-plane strains $(\epsilon_x^0, \epsilon_y^0, \gamma_{xy}^0)$ to in-plane forces (N_x, N_y, N_{xy}). $[D]$ is the bending stiffness matrix that relates curvatures $(\kappa_x, \kappa_y, \kappa_{xy})$ to bending moments (M_x, M_y, M_{xy}). $[B]$ is the bending-extension coupling matrix that relates the in-plane strains to the bending moments and curvatures to in-plane forces. And finally, $[H]$ is the transverse shear stiffness matrix that relates transverse shear strains $(\gamma_{yz}, \gamma_{xz})$ to transverse shear forces (V_y, V_x).The bending effect accounted for by $[B]$ does not exist for homogeneous plates. Actually, if the laminate is symmetric with respect to the middle surface, all the bending-extension coupling coefficients B_{ij} are zero. On the other hand, matrix $[H]$ is only used within the context of first-order shear deformation theory (FSDT) and not within the classical plate theory (CPT) because in the latter transverse shear strains $(\gamma_{yz}, \gamma_{xz})$ are assumed to be zero.

A set of effective laminate engineering material properties (Young's modulus, shear modulus and Poisson's ratios) can be determined from the laminate stiffness matrix, plate thickness and plate bending stiffness of the actual laminate [4].

$$E_x = \frac{A_{11}A_{22} - A_{12}^2}{t \cdot A_{22}} \qquad \text{Eq. 2-43}$$

$$E_y = (A_{11}A_{22} - A_{12}^2)/(t \cdot A_{11}) \qquad \text{Eq. 2-44}$$

$$\nu_{xy} = A_{12}/A_{22} \qquad \text{Eq. 2-45}$$

$$G_{xy} = \frac{A_{66}}{t} \qquad \text{Eq. 2-46}$$

$$G_{yz} = \frac{H_{44}}{t} \qquad \text{Eq. 2-47}$$

$$G_{xz} = \frac{H_{55}}{t} \qquad \text{Eq. 2-48}$$

Summarizing, the Classical Lamination Theory uses the ply unidirectional data, ply thickness, orientation and stacking sequence of the plies to determine the laminate stiffness matrix. The ply modulus matrix is determined from the unidirectional properties for each ply. The stress and strain transformation matrices are determined for each ply and used to transform the modulus terms to the laminate reference axis. Next section is devoted to explain how individual ply elastic properties can be obtained from the actual properties of the composite material components (i.e. fibre and matrix).

2.3.3. Lamina equivalent elastic properties

Glossary

Symbol	Description	Units
$E_{\parallel}$	Longitudinal Young modulus of fibre	N/mm^2
$E_{\perp}$	Transverse Young modulus of fibre	N/mm^2
E_m	Young modulus of the matrix	N/mm^2
G_f	Shear modulus of the fibre	N/mm^2
G_m	Shear modulus of the matrix	N/mm^2
V_f	Volume fraction of fibre in an individual ply	-
ν_f	Fibre's Poisson coefficient	-
ν_m	Matrix Poisson coefficient	-
E_{UD1}	Longitudinal Young's modulus of an unidirectional ply	Pa
E_{UD2}, E_{UD3}	Transverse Young's modulus of an unidirectional ply	Pa
G_{UDij}	Shear moduli of an unidirectional ply	Pa
ν_{UDij}	Poisson coefficients for an unidirectional ply	-
e	Individual layer thickness	m
C_{eq}	Woven balance coefficient for woven roving plies	-
E_{T1}	Young modulus in warp direction of a woven roving lamina	Pa
E_{T2}	Young modulus in weft direction of a woven roving lamina	Pa
E_{T3}	Out-of-plane Young modulus of a woven roving lamina	Pa
G_{Tij}	Shear moduli of a woven roving lamina	Pa
ν_{Tij}	Poisson coefficients of a woven roving lamina	Pa
E_{mat1}, E_{mat2}	In-plane Youn modulus for a strand mat lamina	Pa
E_{mat3}	Out-of-plane Young modulus for a strand mat lamina	Pa
G_{matij}	Shear modulus of a strand mat lamina	Pa
ν_{matij}	Poisson coefficients of a strand mat lamina	Pa

In this section, expressions are presented that can be used to obtain the elastic properties of individual plies from the actual properties of their constituent materials. The formulation presented is in accordance with that reported in [5].

The elastic properties of an unidirectional (UD) lamina are estimated from the properties of its constituents as follows:

$$E_{UD1} = C_{UD1}\left(E_{\parallel} \cdot V_f + E_m \cdot (1 - V_f)\right) \qquad \text{Eq. 2-49}$$

$$E_{UD2} = E_{UD3} = C_{UD2}\left(\frac{E_m}{1-\nu_m^2} \cdot \frac{1 + 0.85 \cdot V_f^2}{(1 - V_f)^{1.25} + \frac{E_m}{E_{\perp}} \cdot \frac{V_f}{1-\nu_m^2}}\right) \qquad \text{Eq. 2-50}$$

$$G_{UD12} = G_{UD13} = C_{UD12} \cdot G_r \cdot \frac{1 + \eta \cdot V_f}{1 - \eta \cdot V_f} \qquad \text{Eq. 2-51}$$

$$G_{UD23} = 0.7 \cdot G_{UD12} \qquad \text{Eq. 2-52}$$

$$\eta = \frac{\left(\frac{G_f}{G_m}\right) - 1}{\left(\frac{G_f}{G_m}\right) + 1} \qquad \text{Eq. 2-53}$$

$$\nu_{UD13} = \nu_{UD12} = C_{UD\nu} \cdot \left[\nu_f \cdot V_f + \nu_m \cdot \left(1 - V_f\right)\right] \qquad \text{Eq. 2-54}$$

$$\nu_{UD21} = \nu_{UD31} = \nu_{UD12} \cdot \frac{E_{UD2}}{E_{UD1}} \qquad \text{Eq. 2-55}$$

$$\nu_{UD32} = \nu_{UD32} = C_{UD\nu} \cdot \left[\nu_f' \cdot V_f + \nu_m \cdot \left(1 - V_f\right)\right] \qquad \text{Eq. 2-56}$$

$$\nu_f' = \nu_f \cdot \frac{E_\perp}{E_\parallel} \qquad \text{Eq. 2-57}$$

In the expressions above, $C_{UD1}, C_{UD2}, C_{UD12}$ and $C_{UD\nu}$ are empirical constants that depend on the actual nature of the fibres being considered. Following reference [5], actual coefficient values for the most common fibre types are summarized in the following table.

	E-Glass	R-Glass	Carbon HS	Carbon IM	Carbon HM	Para-aramid
C_{UD1}	1.00	0.09	1.00	0.85	0.90	0.95
C_{UD2}	0.80	1.20	0.70	0.80	0.85	0.90
C_{UD12}	0.90	1.20	0.90	0.90	1.00	0.55
$C_{UD\nu}$	0.90	0.90	0.80	0.75	0.70	0.90

The elastic properties of for a woven roving lamina are estimated from the properties of its constituents as follows:

$$E_{T1} = \frac{1}{e} \cdot \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right) \qquad \text{Eq. 2-58}$$

$$E_{T2} = \frac{1}{e}\left(A_{22} - \frac{A_{12}^2}{A_{11}}\right) \qquad \text{Eq. 2-59}$$

$$E_{T3} = E_{UD3} \qquad \text{Eq. 2-60}$$

$$G_{T12} = \frac{1}{e} \cdot A_{33} \qquad \text{Eq. 2-61}$$

$$G_{T13} = G_{T23} = 0.9 \cdot G_{T12} \qquad \text{Eq. 2-62}$$

$$\nu_{T12} = A_{12}/A_{22} \qquad \text{Eq. 2-63}$$

$$\nu_{T21} = \nu_{T12} \cdot \left(\frac{E_{T2}}{E_{T1}}\right) \qquad \text{Eq. 2-64}$$

$$\nu_{T32} = \nu_{T31} = \frac{\nu_{UD32} + \nu_{UD31}}{2} \qquad \text{Eq. 2-65}$$

$$\nu_{T13} = \frac{\nu_{UD23} + \nu_{UD13}}{2} \qquad \text{Eq. 2-66}$$

where:

$$A_{11} = e \cdot \left(C_{eq} \cdot Q_{11} + \left(1 - C_{eq}\right) \cdot Q_{22}\right)$$

$$A_{22} = e \cdot \left(C_{eq} \cdot Q_{22} + \left(1 - C_{eq}\right) \cdot Q_{11}\right)$$

$$A_{12} = e \cdot Q_{12}$$

$$A_{33} = e \cdot Q_{33}$$

$$Q_{11} = \frac{E_{UD1}}{1 - \nu_{UD12} \cdot \nu_{UD21}}$$

$$Q_{22} = \frac{E_{UD2}}{1 - \nu_{UD12} \cdot \nu_{UD21}}$$

$$Q_{12} = \frac{\nu_{UD21} \cdot E_{UD1}}{1 - \nu_{UD12} \cdot \nu_{UD21}}$$

$$Q_{33} = G_{UD12}$$

Finally, the elastic properties of a strand mat lamina are estimated from the properties of its constituents as follows:

$$E_{mat1} = E_{mat2} = \frac{3}{8} \cdot E_{UD1} + \frac{5}{8} \cdot E_{UD2} \qquad \text{Eq. 2-67}$$

$$E_{mat3} = E_{UD3} \qquad \text{Eq. 2-68}$$

$$G_{mat12} = \frac{E_{mat1}}{2 \cdot (1 + \nu_{mat21})} \qquad \text{Eq. 2-69}$$

$$G_{mat23} = G_{mat31} = 0.7 \cdot G_{UD12} \quad \text{Eq. 2-70}$$

$$\nu_{mat12} = \nu_{mat21} = \nu_{mat32} = \nu_{mat13} = 0.3 \quad \text{Eq. 2-71}$$

2.4. Serial-Parallel rule of mixtures

In the classical lamination theory presented in section 2.7 individual composite plies are treated as a continuum homogenized material. The laminate stiffness matrix is further evaluated by using the classical lamination theory (CLT). To all effects, the laminated shell is finally treated as an homogeneous orthotropic material.

RamSeries also provides an alternative formulation for FRP laminates in which a serial-parallel (SP) continuum approach is considered [6], [7]. In the SP formulation, the composite material components (namely fibres and matrix) behave as parallel materials in the direction of the fibres alignment and as serial materials in the orthogonal directions. At the same time, each material component is treated taking into account its own constitutive law so that the micro-mechanics of unidirectional composite plies can be assessed. The Serial-Parallel model is further complemented with the application of the rule of mixtures at the level of the laminate to describe the mechanical behaviour of multilayered composite materials [6].

The aim of using the SP model is to assess the mechanical behaviour of the composite as a whole but retaining the individual behaviour of its components (i.e. matrix and fibres). The non-linear behaviour of composite structures due to material degradation is also assessed with the SP model as far as the non-linear behaviour of the constituents is considered in their respective constitutive models. This is not the case in the classic lamination theory in section 2.7 where the material is considered to be linear elastic.

The SP-RoM formulation can be described as follows. First, the heterogeneous composite material is characterized by a composite material domain (c) that can be decomposed into two non-overlapping subdomains associated to the matrix (m) and fibre (f) components respectively:

$$\Omega_c = \Omega_m \cup \Omega_f \quad \text{Eq. 2-72}$$

The matrix and fibre volumetric fractions are denoted by k_m and k_f respectively so that:

$$k_m + k_f = 1 \quad \text{Eq. 2-73}$$

Then, average stress and strain fields can be defined as:

$$\epsilon_c := \frac{\int_{\Omega_c} \epsilon dV}{\int_{\Omega_c} dV}, \epsilon_m := \frac{\int_{\Omega_m} \epsilon dV}{\int_{\Omega_m} dV}, \epsilon_f := \frac{\int_{\Omega_f} \epsilon dV}{\int_{\Omega_f} dV} \quad \text{Eq. 2-74}$$

$$\sigma_c := \frac{\int_{\Omega_c} \sigma dV}{\int_{\Omega_c} dV}, \sigma_m := \frac{\int_{\Omega_m} \sigma dV}{\int_{\Omega_m} dV}, \sigma_f := \frac{\int_{\Omega_f} \sigma dV}{\int_{\Omega_f} dV}$$

And by virtue of the rule of mixtures we can decompose the strain and stress fields as:

$$\epsilon_c = k_m \epsilon_m + k_f \epsilon_f$$

$$\sigma_c = k_m \sigma_m + k_f \sigma_f$$

Eq. 2-75

Regarding the stress as a dependent variable, the material's constitutive law is given by the following set of differential equations:

$$\dot{\sigma}_i = g_i(\epsilon_i, \boldsymbol{\beta}_i, \dot{\epsilon}_i)$$

$$\dot{\boldsymbol{\beta}}_i = h_i(\epsilon_i, \boldsymbol{\beta}_i, \dot{\epsilon}_i)$$

Eq. 2-76

where $i = m, f$ and $\boldsymbol{\beta}$ denotes a vector of internal variables.

Now, the stress and strain fields need to be decomposed in their serial and parallel components. To this aim, appropriate projector tensors are constructed as follows:

$$N_{11} = e_1 \otimes e_1$$

$$P_p = N_{11} \otimes N_{11}$$

$$P_s = I - P_p$$

Eq. 2-77

where $e_1$is the unit vector parallel to the fibre direction, $N_{11}$is the projector second order tensor corresponding to e_1, $P_p$is the parallel fourth order projector tensor and P_s is the serial counterpart.

Hence, the following decomposition can be applied to the composite's stress and strain fields:

$$\epsilon_p = P_p : \epsilon$$

$$\epsilon_s = P_s : \epsilon$$

$$\epsilon = \epsilon_p + \epsilon_s$$

$$\sigma_p = P_p : \sigma$$

$$\sigma_s = P_s : \sigma$$

Eq. 2-78

$$\sigma = \sigma_p + \sigma_s$$

The above decomposition, which is applied to the composite stress and strain fields, are analogously extended to matrix and fibre materials.

Finally, an appropriate closure equation must be devised for long fibre reinforced polymer composites. In this case, the usual and simpler assumption is the one that considers an isostrain condition in the parallel direction and an isostress condition in the serial direction. This results in what is known the basic serial parallel (BSP) closure equations that read as follows:

$$\epsilon_m^p = \epsilon_f^p$$
$$\sigma_m^s = \sigma_f^s \qquad \text{Eq. 2-79}$$

Hence, the governing equations of the serial-parallel problem are the constitutive laws of matrix and fibre materials in Eq. 2-76, the equations that relate average strain and stresses in Eq. 2-75 and the closure equations Eq. 2-79.

The serial-parallel composite solver algorithm implemented in RamSeries as a constitutive law consists in the following steps:

1. Initial approximation of the unknown: the known variables are the converged strains ${}^t\epsilon$and internal variables ${}^t\beta$. The converged strains can be decomposed into its respective fibre and matrix components for the parallel part. And having stored the serial part of the matrix strain ${}^t\epsilon_m^s$(or being zero at the very beginning), the different serial strains can be obtained. To this aim, first the tangent constitutive tensor must be constructed for each composite component:

$$ {}^t\mathbb{C}_i\left({}^t\epsilon_i,\ {}^t\beta_i, \right), i = m, f \qquad \text{Eq. 2-80}$$

   Next, from the tangent constitutive tensor, the following data is computed:

$${}^t\epsilon^s = k_m^t \epsilon_m^s + k_f^t \epsilon_f^s$$

$$\Delta\epsilon^s =^{t+1} \epsilon^s -^t \epsilon^s$$

$$\Delta\epsilon^p =^{t+1} \epsilon^p -^t \epsilon_m^p \qquad \text{Eq. 2-81}$$

$$\mathbb{A} = \left(k_m \mathbb{C}_f^{SS} + k_f \mathbb{C}_m^{SS}\right)^{-1}$$

$$\Delta\epsilon_m^s = \mathbb{A}: \left[\mathbb{C}_f^{SS} \Delta\epsilon^p k_f \left(\mathbb{C}_f^{SP} - \mathbb{C}_m^{PS}\right) \Delta\epsilon^p\right]$$

   Finally, the initial approximation can be obtained as:

$$\epsilon_m^s =^t \epsilon_m^s + \Delta\epsilon_m^s \qquad \text{Eq. 2-82}$$

2. Evaluate the stress residual: to this aim, first the composite component strains must be composed:

$$\epsilon_m = \epsilon_m^p + \epsilon_m^s$$

$$\epsilon_i = \epsilon_i^p + \epsilon_i^s \qquad \text{Eq. 2-83}$$

$$\epsilon_f^s = \frac{1}{k_f}\,{}^{t+1}\epsilon^s - \frac{k_m}{k_f}\epsilon_m^s$$

Next, the stresses can be evaluated for each material component using the corresponding homogeneous constitutive law and the stress residual computed in the following form:

$$\sigma_i(\epsilon_i, \beta_i), i = m, f$$

$$\Delta\sigma^s = \sigma_m^s - \sigma_f^s = P_s : (\sigma_m - \sigma_f) \qquad \text{Eq. 2-84}$$

3. Check convergence:

$$\|\Delta\sigma^s\| > \tau \qquad \text{Eq. 2-85}$$

4. If not converged, update the constitutive tensor and the unknown and return to step 2:

$${}^{t+1}\mathbb{C}_i(\epsilon_i, \beta_i)$$

$$J = \mathbb{C}_m^{SS} + \frac{k_m}{k_f}\mathbb{C}_f^{SS} \qquad \text{Eq. 2-86}$$

$$\epsilon_m^s \coloneqq \epsilon_m^s - J^{-1} : \Delta\sigma^s$$

5. If converged, update the variables for each composite component:

$$\begin{aligned} {}^{t+1}(\epsilon)_i &= {}^{t}(\epsilon)_i \\ {}^{t+1}(\beta)_i &= {}^{t}(\beta)_i \\ {}^{t+1}(\sigma)_i &= {}^{t}(\epsilon)_i \end{aligned} \qquad \text{Eq. 2-87}$$

2.5. The Tsai Wu failure criteria

Of the many failure criteria available, RamSeries implements the Tsai-Wu. This corresponds to a first ply failure analysis for which the sequence of load application is not significant.

First ply failure criteria can be used to assess the global strength of a laminate and is the most common criteria used in linear analysis. Within this context, the structural integrity is usually reported in terms of a reserve factor or margin of safety R. Nevertheless, many of the failure criteria are formulated in terms of an expression that provides a failure index FI, where failure is considered to have occurred if the failure index is equal or greater than unity.

The Tsai Wu failure criteria, like the Hoffman criteria, takes into account the difference in tensile and compressive strengths in the longitudinal and transverse directions of the ply. The failure index is given by [8]:

$$FI = F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + 2F_{12}(\sigma_1\sigma_2) + F_{66}\sigma_{12}^2 + F_1\sigma_1 + F_2\sigma_2 \quad \text{Eq. 2-88}$$

where:

$$F_1 = \left(\frac{1}{S_{t1}}\right) - \left(\frac{1}{S_{c1}}\right) \quad \text{Eq. 2-89}$$

$$F_2 = \left(\frac{1}{S_{t2}}\right) - \left(\frac{1}{S_{c2}}\right) \quad \text{Eq. 2-90}$$

$$F_{11} = \frac{1}{S_{t1}S_{c1}} \quad \text{Eq. 2-91}$$

$$F_{22} = \frac{1}{S_{t2}S_{c2}} \quad \text{Eq. 2-92}$$

$$F_{66} = 1/{T_{12}}^2 \quad \text{Eq. 2-93}$$

$$F_{12} = -0.5(F_{11}F_{22})^{0.5} \quad \text{Eq. 2-94}$$

By replacing the applied stress by the strength (defined as a ratio R times the applied stress) in the failure index expression and equating to unity the following expression is obtained:

$$1.0 = F_{11}R^2\sigma_1^2 + F_{22}R^2\sigma_2^2 + 2F_{12}R^2(\sigma_1\sigma_2) + F_{66}R^2\sigma_{12}^2 + F_1R\sigma_1 + F_2R\sigma_2 \quad \text{Eq. 2-95}$$

The strength ratio Rcan be seen as a safety factor or reserve factor, so that if $R > 1$ the applied stress level is below the strength of the material. On the other hand, if $R < 1$ then failure is predicted.

The above expression can be arranged in quadratic form as:

$$aR^2 + bR + c = 0 \quad \text{Eq. 2-96}$$

where:

$$a = [F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + 2F_{12}(\sigma_1\sigma_2) + F_{66}\sigma_{12}^2] \quad \text{Eq. 2-97}$$

$$b = [F_1\sigma_1 + F_2\sigma_2]$$ Eq. 2-98

$$c = -1.0$$ Eq. 2-99

By solving for R the reserve factor RF is obtained.

2.6. J-2 plasticity theory

The J2 plasticity model is based on the following considerations:

1. Additive decomposition of the strain tensor. In this case one assumes that the strain tensor $\boldsymbol{\epsilon}$ can be decomposed into an elastic and a plastic part $\boldsymbol{\epsilon} = \boldsymbol{\epsilon^e} + \boldsymbol{\epsilon^p}$
2. Elastic stress response. The stress tensor $\boldsymbol{\sigma}$ is related to the elastic strain by means of a stored-energy function W according to the relation $\boldsymbol{\sigma} = \partial W/\partial\boldsymbol{\epsilon}$. In the particular case of linear elasticity, W is a quadratic form of the elastic strain, i.e. $W = \frac{1}{2}\boldsymbol{\epsilon}^e\!:\boldsymbol{C}\!:\boldsymbol{\epsilon}^e$, where $\boldsymbol{C}$ is the elastic stiffness tensor which is assumed to be constant. Then, the stress tensor can be written in the form $\boldsymbol{\sigma} = \boldsymbol{C}\!:(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p)$.
3. The yield condition is defined by means of the so called yield criterion function $f(\boldsymbol{\sigma},\boldsymbol{q})$, where $\boldsymbol{q}$ is a vector of internal variables. The admissible states of the material $\{\boldsymbol{\sigma},\boldsymbol{q}\}$ are constrained as to fulfil the inequality $f(\boldsymbol{\sigma},\boldsymbol{q}) \leq 0$. A typical choice of internal variables for metal's plasticity is $\boldsymbol{q} = \{\boldsymbol{\xi},\boldsymbol{\beta}\}$, where $\boldsymbol{\xi}$ is the equivalent plastic strain that defines the isotropic hardening of the Von Mises yield surface, and $\boldsymbol{\beta}$ defines the center of the Von Mises yield surface in the stress deviatoric space. The resulting J2-plasticity model has the following yield condition:

$$\eta = dev(\sigma) - \beta, tr(\beta) = 0$$ Eq. 2-100

$$f(\boldsymbol{\sigma},\boldsymbol{q}) = \sqrt{\boldsymbol{\eta}\!:\boldsymbol{\eta}} - \sqrt{\frac{2}{3}K(\boldsymbol{\xi})}$$ Eq. 2-101

4. Flow rule and hardening law. The flow rule and hardening law for a J2-plasticity model is given by:

$$d\boldsymbol{\epsilon}^p = \frac{\gamma\boldsymbol{\eta}}{\sqrt{\boldsymbol{\eta}\!:\boldsymbol{\eta}}}$$ Eq. 2-102

$$d\boldsymbol{\beta} = \gamma\frac{2}{3}\boldsymbol{H}(\xi)\frac{\boldsymbol{\eta}}{\sqrt{\boldsymbol{\eta}\!:\boldsymbol{\eta}}}$$ Eq. 2-103

$$d\xi = \gamma\sqrt{\frac{2}{3}}$$ Eq. 2-104

Further explanations can be found in Simo and Hugues (1997).

2.7. Isotropic damage constitutive model

Glossary

Symbol	**Description**	**Units**
d	Damage variable	-
σ	Uniaxial apparent stress	Pa
$\bar{\sigma}$	Uniaxial effective stress	Pa
E	Stiffness modulus	Pa
E^{secd}	Damage secant stiffness modulus	Pa
Ψ	Helmholtz free energy density	J/m^3
Ψ^e	Elastic strain energy	J/m^3
$\boldsymbol{\sigma}$	Apparent Cauchy stress tensor	N/m^2
$\bar{\boldsymbol{\sigma}}$	Effective Cauchy stress tensor	N/m^2
$\boldsymbol{\epsilon}$	Strain tensor	-
$\mathbb{C}^e$	Undamaged elastic tensor	N/m^2
$\mathbb{C}^{secd}$	Damage secant stiffness tensor	N/m^2
τ_σ	Stress energy norm	$N^{1/2}/m$
τ_ϵ	Strain energy norm	$N^{1/2}/m$
$\mathcal{F}$	Damage criterion in stress space	$N^{1/2}/m$
$\mathcal{G}$	Damage criterion in strain space	$N^{1/2}/m$
r	Damage threshold internal variable	$N^{1/2}/m$
q	Hardening/softening internal variable	$N^{1/2}/m$
$\dot{\zeta}$	Damage consistency parameter	$N^{1/2}/m \cdot s$
$\mathcal{H}$	Softening/hardening modulus	$N^{1/2}/m$
r_0	Initial (undamaged material) damage threshold	$N^{1/2}/m$
$\mathcal{H}^d$	Linear hardening/softening parameter	-
q_∞	Saturation equivalent stress for exponential hardening law	$N^{1/2}/m$
A	Exponential parameter for the exponential hardening law	-
G_f	Fracture energy	N/m
τ_σ^I	Energy norm for symmetrical tension/compression damage model	$N^{1/2}/m$
τ_σ^{II}	Energy norm for tension only damage model	$N^{1/2}/m$
τ_σ^{III}	Energy norm for non-symmetrical damage model	$N^{1/2}/m$

The damage model implemented in RamSeries follows the isotropic damage model described in [9], [10] and [11] which is based on the monotonic evolution (no healing effects) of the damage.

2.7.1. One-dimensional isotropic damage model

A simple 1D rheological model can be developed by considering that a material point is subjected to an apparent stress (σ) that acts on an apparent section (s), but due to the existence of micro-defects only the undamaged or effective section ($\bar{s}$) is sustaining the applied load, thus resulting in the existence of an effective stress ($\bar{\sigma}$). Then, by considering the force balance, the following equilibrium equation is obtained:

$$s\sigma = \bar{s}\bar{\sigma} \quad \text{Eq. 2-105}$$

which can be rewritten:

$$\sigma = \left(1 - \frac{s_d}{s}\right)\bar{\sigma} = (1-d)\bar{\sigma} \quad \text{Eq. 2-106}$$

where $s_d$is the damaged section and the dimensionless ratio s_d/srepresents the damage variable denoted by d, whose value must be within the range $0 \leq d \leq 1$.

A simple constitutive equation to describe the one-dimensional isotropic damage model can be stated as follows:

$$\sigma = E^{secd}\epsilon \quad \text{Eq. 2-107}$$

where

$$E^{secd} = (1-d)E \quad \text{Eq. 2-108}$$

is the damage secant stiffness modulus. It can be observed from this relation that the damage variable can be interpreted as a measure of the loss of stiffness modulus of the material. The characteristic stress-strain behaviour of a material with isotropic damage is illustrated in Figure 2-6 in the form of a uniaxial stress-strain curve.

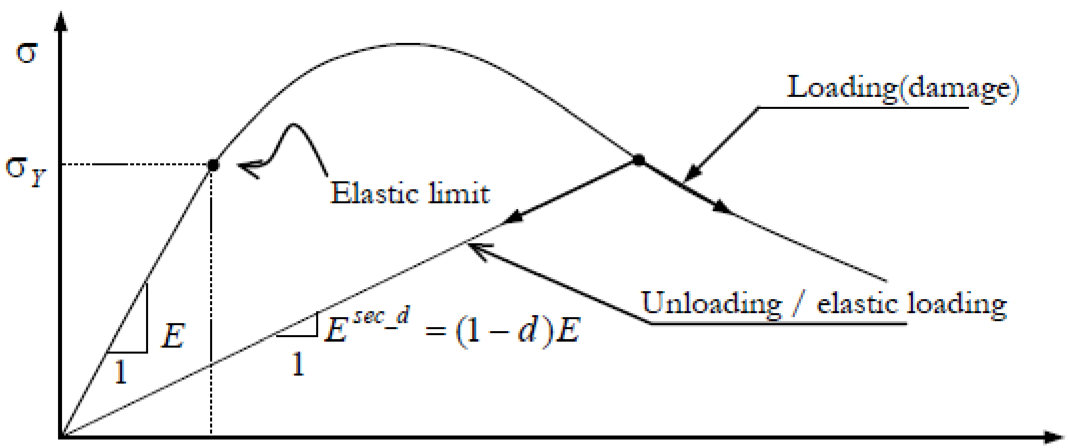

Figure 2-6. One-dimensional stress strain curve characterizing the elastic-isotropic damage behaviour.

The energy equation of the system can be written as follows:

$$\Psi = \frac{1}{2}\epsilon\sigma = (1-d)\frac{1}{2}\epsilon E\epsilon = (1-d)\Psi^e \quad \text{Eq. 2-109}$$

where Ψ^e is the elastic strain energy.

2.7.2. Three-dimensional extension of the isotropic damage model

Note that the damage model described in the previous section depends on a single internal variable, namely the damage parameter d. This implies that the degradation of the material is independent of the orientation so that the damaged elasticity tensor remains isotropic. To extend the isotropic damage model to the more general three-dimensional situation, let us generalize the expression of the Helmholtz free energy in Eq. 2-109 in the following form:

$$\Psi = (1-d)\Psi^e = (1-d)\frac{1}{2}\boldsymbol{\epsilon}:\mathbb{C}^e:\boldsymbol{\epsilon} \qquad \text{Eq. 2-110}$$

where the elastic strain energy $\Psi^e(\epsilon)$ is a function solely of the strain and $\mathbb{C}^e$ is the elasticity tensor of the undamaged material.

By thermodynamic considerations it can be demonstrated that the following expressions must hold:

$$\boldsymbol{\sigma} = \partial\Psi/\,\partial\boldsymbol{\epsilon} \qquad \text{Eq. 2-111}$$

$$\dot{d} \geq 0 \qquad \text{Eq. 2-112}$$

By developing the rate of change of the Helmholtz free energy in Eq. 2-111, a constitutive equation suitable for the isotropic damage model can be obtained.

$$\boldsymbol{\sigma} = \frac{\partial\Psi(\boldsymbol{\epsilon},d)}{\partial\boldsymbol{\epsilon}} = (1-d)\mathbb{C}^e:\boldsymbol{\epsilon} = (1-d)\overline{\boldsymbol{\sigma}} \qquad \text{Eq. 2-113}$$

where $\overline{\boldsymbol{\sigma}}$ is the effective Cauchy stress tensor defined as:

$$\overline{\boldsymbol{\sigma}} = \mathbb{C}^e:\boldsymbol{\epsilon} \qquad \text{Eq. 2-114}$$

From Eq. 2-115 it can be observed that, since the damage parameter is a scalar, the stiffness degradation is isotropic. The stress can be immediately computed once the strain ($\boldsymbol{\epsilon}$) and the damage internal variable (d) are known. And the constitutive equation can be interpreted as the sum of an elastic and an inelastic part.

$$\boldsymbol{\sigma} = (1-d)\mathbb{C}^e:\boldsymbol{\epsilon} = \mathbb{C}^e:\boldsymbol{\epsilon} - d\mathbb{C}^e:\boldsymbol{\epsilon} = \boldsymbol{\sigma}^e - \boldsymbol{\sigma}^i \qquad \text{Eq. 2-115}$$

Finally, Eq. 2-108 that defines the one-dimensional damage secant stiffness modulus can be generalized to the three-dimensional case by defining the elastic-damage stiffness tensor in the following form:

$$\mathbb{C}^{secd} = (1-d)\mathbb{C}^e \qquad \text{Eq. 2-116}$$

Summarizing, the damage constitutive model is completely determined when de damage variable d is known at each time step of the loading/unloading process and when the constitutive equation is supplemented with a damage surface and a damage criterion plus a set of evolution laws for the internals variables and an energy norm of the stress (or strain) tensor. The various elements of the constitutive model listed here are briefly described in the following sections:

Energy norm in the stresses/strain space

A norm in the stress/strain space is a scalar measure that describes the stress/strain state at the current position of the material. A simple norm can be defined in terms of the equivalent stress (τ_σ) and equivalent strain (τ_ϵ) quantities defined as:

$$\tau_\sigma = \sqrt{\boldsymbol{\sigma}: \mathbb{C}^{e^{-1}}: \boldsymbol{\sigma}} \quad \text{Eq. 2-117}$$

$$\tau_\epsilon = \sqrt{\boldsymbol{\epsilon}: \mathbb{C}^{e}: \boldsymbol{\epsilon}} \quad \text{Eq. 2-118}$$

$$\tau_\sigma = (1-d)\tau_\epsilon \quad \text{Eq. 2-119}$$

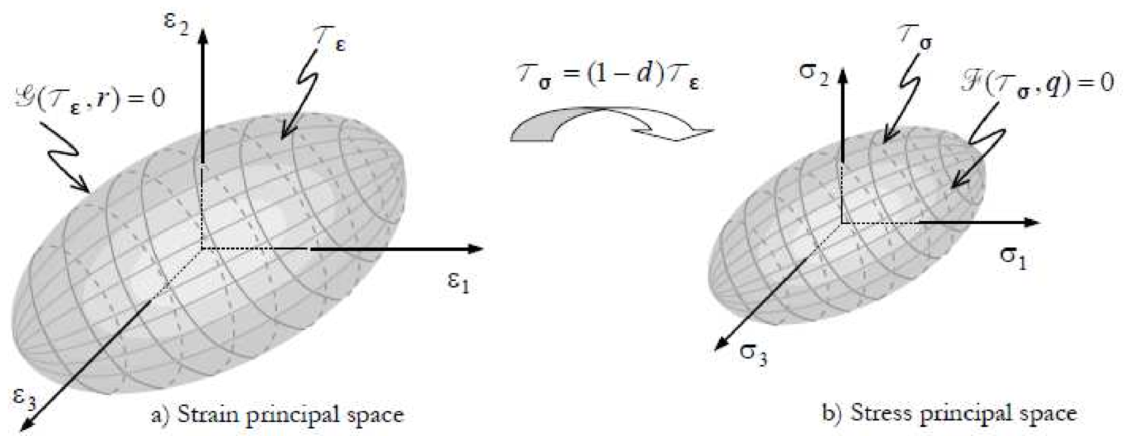

Figure 2-7. Damage surface in the strain (a) and stress (b) spaces that defines the damage criteria in terms of the stress/strain state at each point of the material.

Damage criterion

The damage criterion can be defined either in stress or strain space as follows:

$$\mathcal{F}(\tau_\sigma, q) = \tau_\sigma - q(r) \le 0 \quad \text{Eq. 2-120}$$

$$\mathcal{G}(\tau_\epsilon, r) = \tau_\epsilon - r \le 0 \quad \text{Eq. 2-121}$$

where r is an internal variable than can be interpreted as the current damage threshold and q is a function of r defining a hardening/softening variable. Both r and q are related to the damage variable d. The damage criterion given by Eq. 2-120 implies that the stress state must be on or inside the damage surface. In the case of the stress state lying inside the damage surface, the material exhibits elastic behaviour either in a loading or unloading condition. On the contrary, in the case of the stress state lying on the damage surface, the material will behave elastically or continue increasing the damage depending on the actual loading condition.

From Eq. 2-121 , it is observed that the undamaged material initial value (r_0) of the damage threshold determines the initial yield in the strain space so that the material starts fail when the strain energy norm exceeds the value r_0. From a uniaxial test the value of r_0 can be related to the yield stress measured in the laboratory:

$$r_0 = \frac{\sigma_Y}{\sqrt{E}} \quad \text{Eq. 2-122}$$

Evolution law

The evolution laws for the yield limit (r) and for the damage variable (d) are:

$$\dot{r} = \dot{\zeta}(\boldsymbol{\epsilon}, r) \quad \text{Eq. 2-123}$$

$$\dot{d} = \dot{\zeta}\mathcal{H}(\tau_{\boldsymbol{\epsilon}}, d) \quad \text{Eq. 2-124}$$

where $\dot{\zeta}$ is the damage consistency parameter and $\mathcal{H}$ is the continuum softening/hardening modulus.

Linear hardening/softening law

For the linear hardening/softening law it is assumed that q has the following linear relation with r:

$$q(r) = \begin{cases} r_0 & r \leq 0 \\ r_0 + \mathcal{H}^d(r - r_0) \nless 0 & r > r_0 \end{cases} \quad \text{Eq. 2-125}$$

so that the hardening law is fully characterized by the value of the initial damage threshold r_0 and the value of the hardening (softening) parameter $\mathcal{H}^d$ that, as shown in the figure, corresponds to the slope of the linear hardening/softening relation. As expected, positive values of $\mathcal{H}^d$ correspond to hardening behaviour while negative values correspond to softening.

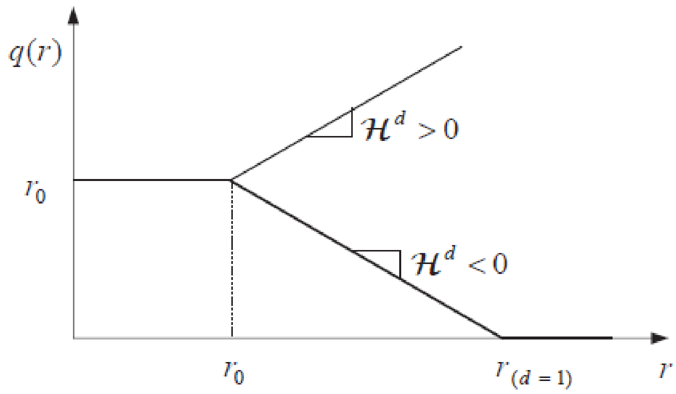

Figure 2-8. Schematic representation of the linear hardening/softening law

Exponential hardening/softening law

In the case of an exponential hardening/softening law, the relation of q with r can be expressed as follows:

$$q(r) = q_\infty - (q_\infty - r_0)e^{A\left(1-\frac{r}{r_0}\right)} \qquad \text{Eq. 2-126}$$

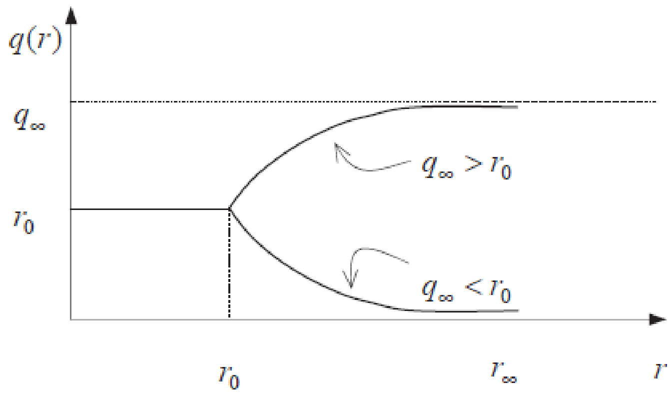

Figure 2-9. Schematic representation of the exponential hardening/softening law

The hardening law is again fully characterized by two scalar values, the saturation equivalent stress q_∞ and the exponential factor A which is related to the fracture energy G_f of the material by the following expression [12]:

$$A = \left(\frac{G_f}{r_0^2 \cdot l^*} - \frac{1}{2}\right)^{-1} \qquad \text{Eq. 2-127}$$

Summary of the isotropic damage model

The fundamental ingredients of the isotropic damage model described in the previous sections can be summarized, as it is done in [11], assuming a small deformation approximation and adopting a description where the strain is the free variable.

Helmholtz free energy	$\Psi(\boldsymbol{\epsilon}, r) = \big(1 - d(r)\big)\Psi^{\mathrm{e}}\,, \text{with } \Psi^{\mathrm{e}} = \frac{1}{2}(\boldsymbol{\epsilon}:\mathbb{C}^{e}:\boldsymbol{\epsilon})$	Eq. 2-128
Damage parameter	$d(r) = 1 - \frac{q}{r}\,; q \in [r_0, a], a \neq \infty\,; d \in [0,1]$	Eq. 2-129
Constitutive equations	$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\epsilon}} = (1-d)\overline{\boldsymbol{\sigma}} = (1-d)\mathbb{C}^{e}:\boldsymbol{\epsilon}$	Eq. 2-130
Evolution law	$\dot{r} = \zeta\,; \begin{cases} r \in [r_0, \infty) \\ r_0 = r\vert_{t=0} = \frac{\sigma_Y}{\sqrt{E}} \end{cases}$	Eq. 2-131
Damage criterion	$\mathcal{G}(\boldsymbol{\epsilon}, r) = \tau_{\epsilon} - r = \sqrt{\boldsymbol{\epsilon}:\mathbb{C}^{e}:\boldsymbol{\epsilon}} - r$	Eq. 2-132
Hardening law	$\dot{q} = \mathcal{H}^{d}(r)\dot{r}; \mathcal{H}^{d} = q'(r) \leq 0$	Eq. 2-133

2.7.3. Energy norms

The actual behaviour of the damaged material is strongly influenced by the yield damage surface but also by the actual energy norm under consideration. Three different energy norms are implemented in RamSeries, which are commonly used together with the isotropic damage model.These are briefly described in what follows.

Symmetrical tension-compression damage model (mode I)

This model can be used when the material behaves the same under tension or compression. In this case the energy norm can be expressed as:

$$\tau_{\sigma}^{I} = \sqrt{\boldsymbol{\sigma}:\mathbb{C}^{e^{-1}}:\boldsymbol{\sigma}} = (1-d)\sqrt{\overline{\boldsymbol{\sigma}}:\boldsymbol{\epsilon}} \qquad \text{Eq. 2-134}$$

where l^{*} is the characteristic length of the finite element.

For the sake of illustration, it is useful to consider a plane stress condition, in which case the damage surface is represented by an ellipse. By considering the mode I energy norm, we are assuming the stress limit σ_Y to be the same in tension and compression. Additionally, the damage surface is assumed to evolve symmetrically as shown in the following figure.

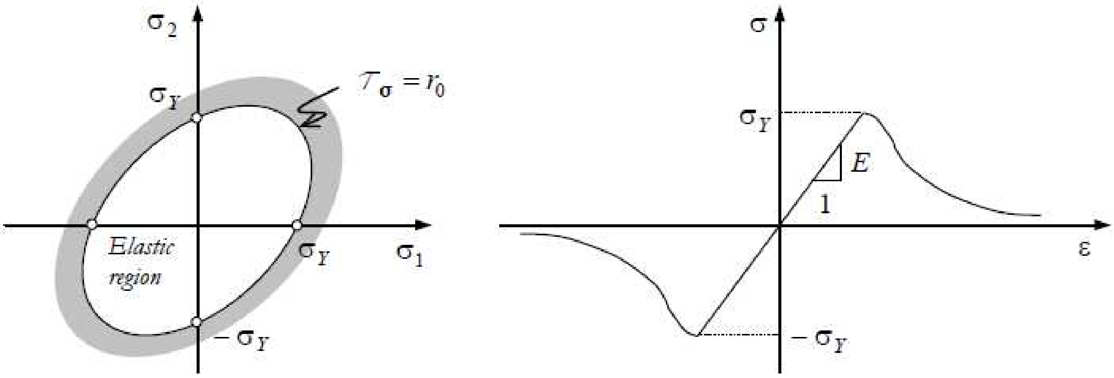

Figure 2-10. Schematic representation of damage model's Mode I energy norm. (a) Damage surface corresponding to a plane stress condition. (b) Uniaxial stress-strain curve in tension and compression.

Tension-only damage model (Mode II)

In this model, damage is assumed to occur only in tension. Under compression, the material will always behave elastically and will never fail. In this case, the energy norm can be defined as:

$$\tau_\sigma^{II} = \sqrt{\boldsymbol{\sigma}^+ : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} \qquad \text{Eq. 2-135}$$

where

$$\boldsymbol{\sigma}^+ = \frac{\boldsymbol{\sigma} + |\boldsymbol{\sigma}|}{2} \qquad \text{Eq. 2-136}$$

and the relationship between real and effective stress still holds:

$$\boldsymbol{\sigma}^+ = (1 - d)\overline{\boldsymbol{\sigma}}^+ \qquad \text{Eq. 2-137}$$

As in the previous section, it is useful to represent the Mode II damage surface corresponding to a plane stress condition and the uniaxial stress-strain curve for the corresponding material.

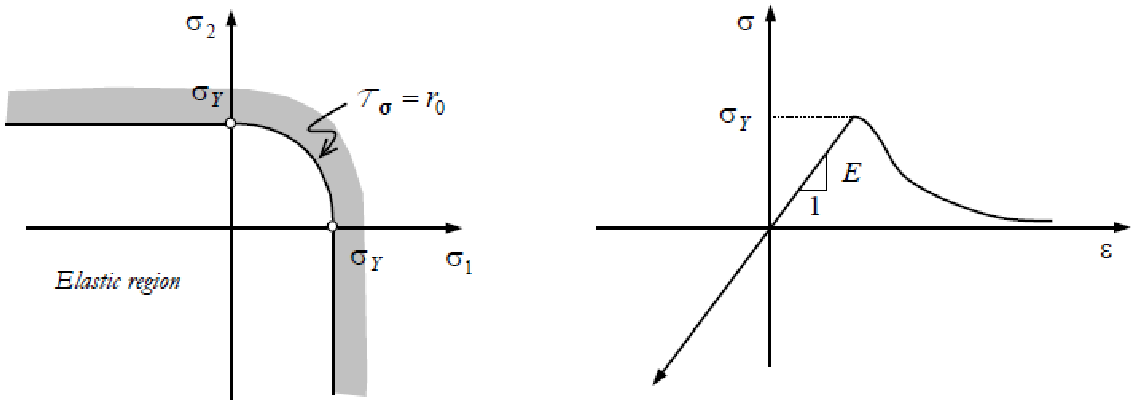

Figure 2-11. Schematic representation of damage model's Mode II energy norm. (a) Damage surface corresponding to a plane stress condition. (b) Uniaxial stress-strain curve in tension and compression.

Non-symmetrical damage model

Finally, Mode III energy norm can be used to simulate the mechanical response of materials whose behaviour under tension differs from that I n compression. A typical application of this kind of energy norm is to model materials like concrete which exhibit quite larger strength in compression than in tension. The following energy norm can be used in these cases:

$$\tau_\sigma^{III} = \left(\theta + \frac{1-\theta}{n}\right)\sqrt{\boldsymbol{\sigma}:\mathbb{C}^{e^{-1}}:\boldsymbol{\sigma}} \quad \text{Eq. 2-138}$$

where θ is a weight factor that depends on the actual stress state as follows:

$$\theta = \frac{\sum_{i=1}^{3}\langle\sigma_i\rangle}{\sum_{i=1}^{3}|\sigma_i|} \quad \text{Eq. 2-139}$$

and the parameter$n$ is defined as the ratio between the compression and the traction elastic limits.

$$n = \frac{\sigma_Y^c}{\sigma_Y^t} \quad \text{Eq. 2-140}$$

The Macaulay bracket $\langle\sigma_i\rangle$ in the above expression is defined as:

$$\langle\sigma_i\rangle = \frac{\sigma_i + |\sigma_i|}{2} \quad \text{Eq. 2-141}$$

The schematic representation of Mode III damage model is shown in the following figure.

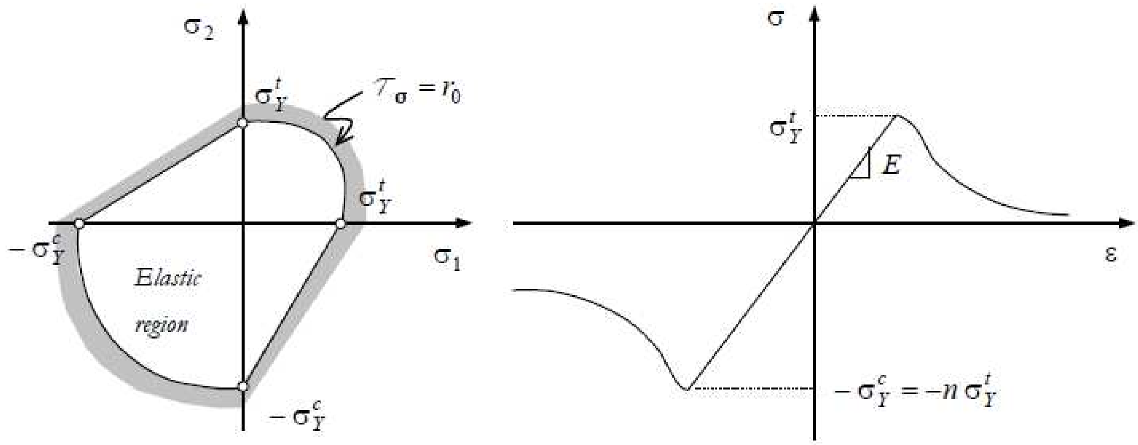

Figure 2-12. Schematic representation of damage model’s Mode III energy norm. (a) Damage surface corresponding to a plane stress condition. (b) Uniaxial stress-strain curve in tension and compression.

3. Dynamic analysis

In this section a brief summary of the theory of dynamic analysis of structures is provided.

3.1. Direct integration

The direct integration methods seek the time history of the dynamic response. Such a response is not obtained in continuous form but rather in a predetermined discrete series of points in time t_i. One of the most popular direct integration methods is the Newmark method. The starting point of the problem is the governing equation of a structure with various degrees of freedom. Such a governing equation can be written in the following form:

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{C}\dot{\boldsymbol{D}} + \boldsymbol{K}\boldsymbol{D} = \boldsymbol{P}(t) \qquad \text{Eq. 3-1}$$

For time $t = t_i$, this equation is discretised as follows:

$$\boldsymbol{M}\ddot{\boldsymbol{D}}_{\mathrm{i+1}} + \boldsymbol{C}\dot{\boldsymbol{D}}_{\mathrm{i+1}} + \boldsymbol{K}\boldsymbol{D}_{\mathrm{i+1}} = \boldsymbol{P}_{\mathrm{i}} \qquad \text{Eq. 3-2}$$

On the other hand, the velocity and the acceleration are expressed as follows:

$$\dot{\boldsymbol{D}}_{\mathrm{i+1}} = \frac{\gamma}{\beta\Delta t}[\boldsymbol{D}_{\mathrm{i+1}} - \boldsymbol{D}_{\mathrm{i}}] + \left(1 - \frac{\gamma}{\beta}\right)\dot{\boldsymbol{D}}_{\mathrm{i}} + \left(1 - \frac{\gamma}{2\beta}\right)\Delta t\ddot{\boldsymbol{D}}_{\mathrm{i}} \qquad \text{Eq. 3-3}$$

$$\ddot{\boldsymbol{D}}_{\mathrm{i+1}} = \frac{1}{\beta\Delta t^2}[\boldsymbol{D}_{\mathrm{i+1}} - \boldsymbol{D}_{\mathrm{i-1}} - \boldsymbol{D}_{\mathrm{i}}\Delta t] - \left(\frac{1}{2\beta} - 1\right)\ddot{\boldsymbol{D}}_{\mathrm{i}} \qquad \text{Eq. 3-4}$$

Substituting these expressions into the equation of motion, we obtain:

$$\boldsymbol{K}^{\mathrm{c}}\boldsymbol{D}_{\mathrm{i+1}} = \boldsymbol{P}^{\mathrm{c}}_{\mathrm{i+1}} \qquad \text{Eq. 3-5}$$

where,

$$\boldsymbol{K}^{\mathrm{c}} = \boldsymbol{K} + \frac{1}{\beta\Delta t^2}\boldsymbol{M} + \frac{\gamma}{\beta\Delta t}\boldsymbol{C} \qquad \text{Eq. 3-6}$$

$$\begin{aligned}\boldsymbol{P}^{\mathrm{c}}_{\mathrm{i+1}} = \boldsymbol{P}_{\mathrm{i+1}} &+ \boldsymbol{M}\left[\frac{1}{\beta\Delta t^2}\boldsymbol{D}_{\mathrm{i}} + \frac{1}{\beta\Delta t}\dot{\boldsymbol{D}}_{\mathrm{i}} + \left(\frac{1}{2\beta} - 1\right)\ddot{\boldsymbol{D}}_{\mathrm{i}}\right] \\ &+ \boldsymbol{C}\left[\frac{\gamma}{\beta\Delta t}\boldsymbol{D}_{\mathrm{i}} + \left(\frac{\gamma}{\beta} - 1\right)\dot{\boldsymbol{D}}_{\mathrm{i}} + \left(\frac{\gamma}{2\beta} - 1\right)\Delta t\ddot{\boldsymbol{D}}_{\mathrm{i}}\right]\end{aligned} \qquad \text{Eq. 3-7}$$

Usually, the initial conditions to close the problem consist on assuming that the structure has neither displacements nor velocity.

3.2. Modal analysis

Eq. 3-1 represents the system of equations corresponding to a structure with n degrees of freedom. The corresponding free vibrations not damped problem is represented by the following equation:

$$\boldsymbol{M}\ddot{\boldsymbol{D}} + \boldsymbol{K}\boldsymbol{D} = 0 \qquad \text{Eq. 3-8}$$

Assuming the displacement vector is going to have a sinusoidal response in time, the admissible solutions to the above problem will be a linear combination of solutions to the generalized eigenvalue problem:

$$\omega^2\boldsymbol{M}\boldsymbol{D} = -\boldsymbol{K}\boldsymbol{D} \qquad \text{Eq. 3-9}$$

where ω^2 is the eigenvalue. Hence, by solving a classic eigenvalue problem, the natural vibration modes of the structure can be found. The solution eigenvectors, which are orthogonal to both the mass and the stiffness matrix, form a complete base of the displacements vector field. Hence, free vibration displacements can be written as a linear combination of the modal vectors $\boldsymbol{\varphi}_i$:

$$\boldsymbol{D} = \sum_{i=1}^{n} \boldsymbol{\varphi}_i y_i(t) \qquad \text{Eq. 3-10}$$

where $y_i(t)$ are scalar functions of time, called generalized coordinates.

Using the previous expressions and taking into account the orthogonality conditions, we can transform the governing equation as to obtain the following system of n equations with one degree of freedom each:

$$\boldsymbol{M}_j^*\ddot{y}_i(t) + \boldsymbol{C}_j^*\dot{y}_j(t) + \boldsymbol{K}_j^* y_j(t) = \boldsymbol{\varphi}_i^T\boldsymbol{P}(t) \qquad \text{Eq. 3-11}$$

where:

$$\boldsymbol{M}_j^* = \boldsymbol{\varphi}_j^T\boldsymbol{M}\boldsymbol{\varphi}_j \qquad \text{Eq. 3-12}$$

$$\boldsymbol{C}_j^* = \boldsymbol{\varphi}_j^T\boldsymbol{C}\boldsymbol{\varphi}_j \qquad \text{Eq. 3-13}$$

$$\boldsymbol{K}_j^* = \boldsymbol{\varphi}_j^T\boldsymbol{K}\boldsymbol{\varphi}_j \qquad \text{Eq. 3-14}$$

3.3. Spectral analysis

Spectrum analysis is typically used in RamSeries to solve dynamic problems associated to seismic actions that enforce the movement at the base of the foundation of a given structure. To this aim, the decoupled equation of motion for each mode is expressed as:

$$\boldsymbol{M}_j^* \ddot{y}_i(t) + \boldsymbol{C}_j^* \dot{y}_j(t) + \boldsymbol{K}_j^* y_j(t) = -\frac{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{J}}{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{\varphi}_j} a(t) \qquad \text{Eq. 3-15}$$

where $a(t)$is the seismic acceleration. This equation can be solved using the response spectra. In this case, only the maximum response of the structure is obtained taking into account the maximum acceleration $|a(t)|_{max} = S$. In this case, it is evident that the maximum response acceleration of the system would amount to:

$$|y_i(t)|_{max} = \frac{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{J}}{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{\varphi}_j} S \qquad \text{Eq. 3-16}$$

As a consequence, the maximum displacement is given by:

$$|y_i(t)|_{max} = \frac{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{J}}{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{\varphi}_j} \cdot \frac{S_j}{\omega_j^2} \qquad \text{Eq. 3-17}$$

Using this solution, we can calculate the maximum displacements in all the nodes of the discrete structure for mode j in the following form:

$$\boldsymbol{D}_{max}^j = \begin{bmatrix} D_1^j \\ D_2^j \\ \cdots \\ D_n^j \end{bmatrix} = \boldsymbol{\varphi}_j \left|y_j(t)\right|_{max} = \boldsymbol{\varphi}_j \frac{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{J}}{\boldsymbol{\varphi}_j^T \boldsymbol{M} \boldsymbol{\varphi}_j} \cdot \frac{S_j}{\omega_j^2} = \boldsymbol{A}_j \cdot \frac{S_j}{\omega_j^2} \qquad \text{Eq. 3-18}$$

where $\boldsymbol{A}$ is the vector of the modal participation coefficients corresponding to mode j of the vibration. Supposing that for each degree of freedom the maximum response does not occur at the same instant in each mode, the maximum response of the structure will not be equal to the sum of the maximum corresponding to each mode:

$$\boldsymbol{D}_{max} \neq \sum_{i=1}^{n} \boldsymbol{D}_j \qquad \text{Eq. 3-19}$$

Different formulas to find the value of $\boldsymbol{D}$through $\boldsymbol{D}$have been proposed. The most simple and, at the same time, most used is that which establishes that the response is equal to the square root of the sum of the squares of the modal responses:

$$\boldsymbol{D}_{max} = \sqrt{\sum_{i=1}^{q} (\boldsymbol{D}_{max}^j)^2} \qquad \text{Eq. 3-20}$$

As for the stresses, reactions and in general any response R which is to be determined, we can get analogously:

$$\boldsymbol{R}_{max} = \sqrt{\sum_{i=1}^{q} (\boldsymbol{R}_{max}^{j})^2} \qquad \text{Eq. 3-21}$$

4. Non-linear analysis

This section provides a summary of the non-linear theories of solids and structures considered in RamSeries. Several aspects concerning the solution of a non-linear system of equations are addressed. Typical non-linear constitutive models for metals, like the J-2 plasticity theory, are also explained.

4.1. Non-linear systems of equations

All strategies for the resolution of a non-linear system of equations are related with the decomposition of the non-linear problem in several linear systems each of them solved using the well-established methods for the solution of linear problems. A well-known of these strategies is the Newton-Raphson method. It consists on an incremental-iterative scheme for which the total applied load is decomposed into several increments. For each increment, an iterative procedure is performed until the desired convergence criteria is achieved. There are several methods to control the fulfilment of the load-displacement curve of the structure. Each one of these methods is actually based on the control of a different parameter, like for instance the load, the displacement or the arc-length increments. Some advanced procedures, like the line-search, automatic incrementation or automatic arc-length switch can be applied to speed-up the calculations.

Deeper explanations can be found in [13] and [14].

5. References

[1] E. Oñate Ibáñez de Navarra and Centro Internacional de Métodos Numéricos en Ingeniería, *Cálculo de estructuras por el método de elementos finitos: análisis estático lineal*. Barcelona: CIMNE, 1995.

[2] L. E. Malvern, *Introduction to the mechanics of a continuous medium*. Englewood Cliffs, N.J: Prentice-Hall, 1969.

[3] P. Hopkins, "Benchmarks for Membrane and bending analysis of Laminated Shells. Part 1. Stiffness matrix and thermal characteristics.," NAFEMS, 2005.

[4] E. J. Barbero, *Introduction to composite materials design*, Third edition. Boca Raton: CRC Press, Taylor & Francis Group, CRC Press is an imprint of the Taylor & Francis Group an informa business, 2018.

[5] Bureau Veritas, "Hull in composite materials and plywood, material approval, design principles, construction and survey," Bureau Veritas, Feb. 2017.

[6] F. Rastellini, S. Oller, O. Salomón, and E. Oñate, "Composite materials non-linear modelling for long fibre-reinforced laminates," *Comput. Struct.*, vol. 86, no. 9, pp. 879–896, May 2008.

[7] Xavier Martínez, "Micro mechanical simulation of composite materials using the serial/parallel mixing theory," Structural analysis, Universitat Politècnica de Catalunya, Barcelona, 2008.

[8] P. Hopkins, "Benchmarks for Membrane and bending analysis of Laminated Shells. Part 2. Strength Analysis.," NAFEMS, 2005.

[9] J. C. Simo and J. W. Ju, "Strain- and stress-based continuum damage models—I. Formulation," *Int. J. Solids Struct.*, vol. 23, no. 7, pp. 821–840, 1987.

[10] E. W. V. Chaves and X. Oliver, *A three dimensional setting for strong discontinuities modelling in failure mechanics*, 1. ed. Barcelona: International Center for Numerical Methods in Engineering, 2003.

[11] E. W. . Chaves and J. Oliver, "Notes on continuum damage models," 2013.

[12] J. Oliver, S. Oller, and M. Cervera, "Isotropic damage models and smeared crack analysis of concrete," 1990.

[13] K.-J. Bathe, *Finite element procedures*. S.l.: s.n., 2014.

[14] M. A. Crisfield, *Non-linear finite element analysis of solids and structures. Advanced topics.*, vol. 2. Chichester ; New York: Wiley, 1991.